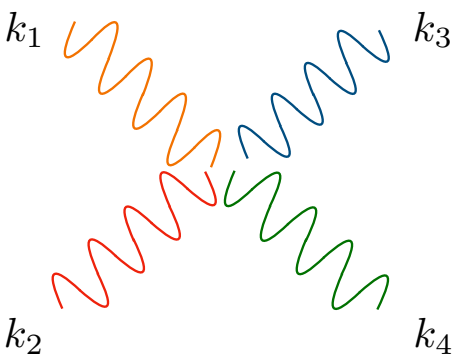
Wave kinetic theory for the forced/ dissipated NLS equation

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Joint work with **Zaher Hani**
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Combinatorics from singular SPDEs to kinetic equations *IECL - Nancy*

Statistical physics of wave systems



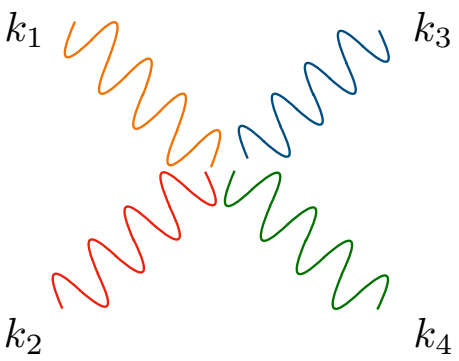
$k_1 + k_2 = k_3 + k_4$

$u_k(0)$ random and independent

$$\frac{\partial u_k}{\partial t} = -i\frac{\partial H}{\partial \overline{u}_k}$$

e.g. $$H = \sum_k w(k)|u_k|^2 + \lambda \sum_{k_1+k_2=k_3+k_4} u_{k_1}\overline{u}_{k_2}u_{k_3}\overline{u}_{k_4}$$

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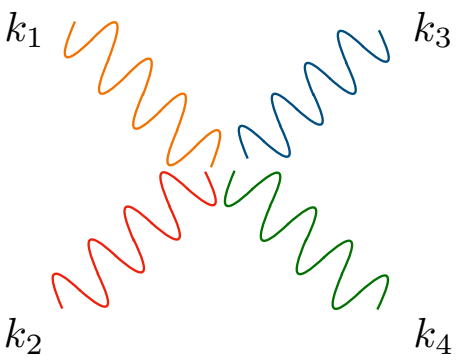
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Goal: understand the average behavior of the system as $L \to \infty$ and $\lambda \to 0$

How does the probability measure at $t = 0$ evolve under nonlinear dynamics?

e.g. $\mathbb{E}|u_k(t)|^{2p}, \qquad p \in \mathbb{N}$

correlations, ...

Wave Kinetic Equation for NLS

$$i\partial_t u + \Delta u = \lambda |u|^2 u \qquad x \in \mathbb{T}^d_L = \mathbb{R}^d/(L\mathbb{Z}^d) \qquad d \geq 3$$

$$u(t,x)|_{t=0} = L^{-d/2} \sum_{k \in \mathbb{Z}^d_L} c_k \, \eta_k \, e^{ik\cdot x} \qquad c_k = c(k) \qquad c : \mathbb{R}^d \to [0,\infty) \qquad c \in \mathcal{S}(\mathbb{R}^d)$$

- Classical assumptions of random coefficients η_k: $\mathbb{E}\eta_k = 0 \qquad \mathbb{E}|\eta_k|^2 = 1$

  RP: $\eta_k = e^{i\phi_k}$ with $\phi_k \sim \text{unif}(0, 2\pi)$, independent phases

  RPA: $\eta_k = R_k \, e^{i\phi_k}$ with $R_k \geq 0$ independent amplitudes and $\phi_k \sim \text{unif}(0, 2\pi)$ independent phases

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- Size of initial data $$\mathbb{E}|u(0,x)|^2 = L^{-d} \sum_{k \in \mathbb{Z}_L^d} c_k^2 \longrightarrow \int_{\mathbb{R}^d} c(k)^2 \, dk = \mathcal{O}(1) \qquad \text{as } L \to \infty$$

Wave Kinetic Equation for NLS

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- **Kinetic timescale:** $T_{\text{kin}} = \lambda^{-2} \gg T_{\text{nl}} = \lambda^{-1}$ (nonlinear timescale) [Deng-Hani]

- **Second order moments:** $\mathbb{E}|u_k(t)|^2 \approx n\left(\frac{t}{T_{\text{kin}}}, k\right)$ satisfy the following kinetic equation

$$\partial_t n = \mathcal{K}(n) \qquad k \in \mathbb{R}^d$$
$$n(t,k)|_{t=0} = c(k)^2$$

$\lambda \to 0$
$L \to \infty$
scaling laws

$$\mathcal{K}(n)(t,k) = \int_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ k = k_1 - k_2 + k_3}} \delta\left(|k_1|^2 - |k_2|^2 + |k_3|^2 - |k|^2\right) \left[n_1 n_2 n_3 - n n_2 n_3 + n n_1 n_3 - n n_1 n_2\right] dk_1 \, dk_3$$

$$n_j = n(t,k_j)$$

Wave Kinetic Equations: other systems

- Linear Schrödinger with random potential [Erdös-Salmhofer-Yau 2008]
- Cubic NLS on lattice on statistical equilibrium [Lukkarinen-Spohn 2011]
- Inhomogeneous version of WKE with $(\partial_t + k \cdot \nabla_x) n(t, x, k)$ using localized wave packets in $\mathbb{R}^d$ [Hani-Shatah-Zhu 2023]
- Random dynamics: Schrödinger-type equation on $\mathbb{C}^{2N+1}$ with random matrix instead of Δ [Dubach-Germain-Harrop-Griffiths 2023]
- KdV-type equation with a time-dependent Stratonovich stochastic forcing that randomizes the phases [Staffilani-Tran 2021]
- Semilinear Klein-Gordon system [de Suzzoni-Stingo-Touati 2025]
- FPUT and Majda-McLaughlin-Tabak equation [Vassilev 2025], [Vassilev-Wu 2026]

Stationary solutions and cascades

- Well-posedness of WKE studied by [Escobedo-Velázquez (2015)] [Germain-Ionescu-Tran (2018)]

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Stationary solutions and energy cascade

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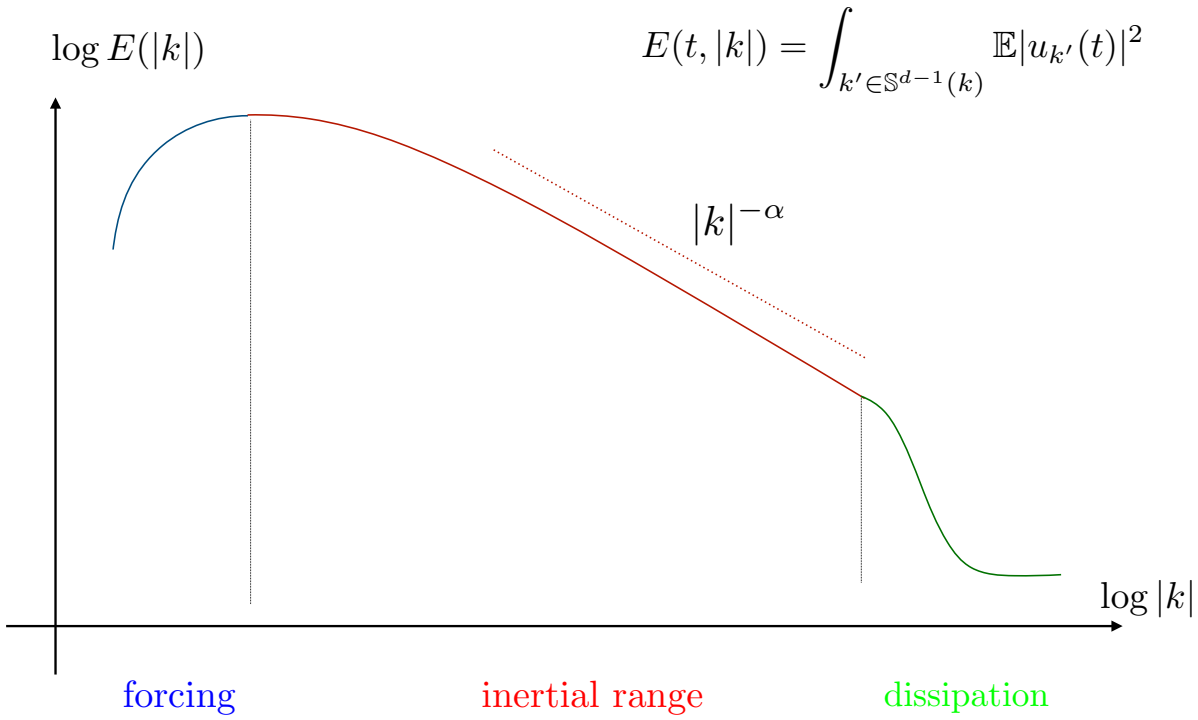
- Zakharov and his collaborators found stationary "solutions" of the form $n(k) = |k|^{-\alpha}$ believed to be connected to cascades.

Kolmogorov-Zakharov spectra

Problem: Hard to make sense of notion of solution due to singularity at $k = 0$

Stationary solutions and energy cascade

- Forcing injects energy on large scales
- Nonlinear wave interactions transfer energy from large to smaller scales
- Energy is dissipated at very small scales
- Scale separation: forcing and dissipation are negligible on an intermediate range of scales



The Zakharov - L'vov model

- We would like to have a model that accounts for energy injection, damping, propagation and interactions.

$$\frac{\partial u_k}{\partial t} = -i\frac{\partial H}{\partial \overline{u}_k} + f_k(t) - \gamma_k u_k$$

Zakharov-L'vov model (1975)

$f_k(t)$ random forcing supported on $|k| \lesssim 1$

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- We consider the Wick-ordered cubic NLS Hamiltonian, leading to

$$\partial_t u = -i\Delta u + i\lambda \left(|u|^2 - \frac{2}{L^d}\int_{\mathbb{T}^d_L} |u|^2 \right) u - \nu_1 (1-\Delta)^r u + \nu_2 \, \dot{\beta} \qquad \begin{array}{c} x \in \mathbb{T}^d_L \\ r \in (0,1],\ \lambda, \nu_1, \nu_2 > 0 \end{array}$$

$$\dot{\beta}(t,x) = L^{-d/2} \sum_{k \in \mathbb{Z}^d_L} b_k \, e^{ik\cdot x} \, d\beta_k(t)$$

$\{\beta_k(t)\}_{k\in\mathbb{Z}^d_L}$ ind. std. complex Wiener processes

$b : \mathbb{R}^d \to [0,\infty)$ Schwartz

$$u(t,x)|_{t=0} = L^{-d/2} \sum_{k \in \mathbb{Z}^d_L} c_k \, \eta_k \, e^{ik\cdot x}$$

$\{\eta_k\}_{k\in\mathbb{Z}^d_L}$ i.i.d. std. complex Gaussians

$c : \mathbb{R}^d \to [0,\infty)$ Schwartz

The Zakharov - L'vov model

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Goals:

1) Model closer to physical setting: understand role of forcing and dissipation

2) Does the kinetic description of such a wave system depend on the way randomness is introduced?

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2) Does the kinetic description of such a wave system depend on the way randomness is introduced?

3) Falkovich: "The basic question is that of universality: to what extent the statistics of such fluctuations is independent of the details of external forcing and internal friction and which features are common to different turbulent systems."

A priori estimates on energy

$$\partial_t u = -i\Delta u + i\lambda \left(|u|^2 - \frac{2}{L^d} \int_{\mathbb{T}_L^d} |u|^2 \right) u - \nu_1 (1-\Delta)^r u + \nu_2 \, \dot{\beta} \qquad x \in \mathbb{T}_L^d$$

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- Itô's formula implies

$$\mathbb{E} \, \|u(t)\|^2_{L^2(\mathbb{T}_L^d)} - \mathbb{E} \, \|u(0)\|^2_{L^2(\mathbb{T}_L^d)} = -2\nu_1 \, \mathbb{E} \int_0^t \left\| (1-\Delta)^{r/2} u(t') \right\|^2_{L^2(\mathbb{T}_L^d)} dt' + 2\nu_2^2 \, t \sum_{k \in \mathbb{Z}_L^d} b_k^2$$

Must impose $\nu_1 \sim \nu_2^2$ so that forcing and dissipation have the same order of magnitude (w.r.t. the size of the box!)

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- We define the *forcing timescale* $T_{\text{for}} = \nu^{-1}$

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Goal: Obtain kinetic description as $\lambda \to 0$
$\nu \to 0$
$L \to \infty$

Kinetic theory for Stochastic NLS

$$\partial_t u = -i\Delta u + i\lambda \left(|u|^2 - \frac{2}{L^d} \int_{\mathbb{T}^d_L} |u|^2 \right) u - \nu_1 (1-\Delta)^r u + \nu_2 \dot{\beta} \qquad x \in \mathbb{T}^d_L$$

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Theorem: (G. - Hani, 2024)

Let $d \geq 2$, $r \in (0,1]$, and $b, c \in \mathcal{S}(\mathbb{R}^d; [0,\infty))$ be Schwartz functions. Let $T_{\text{for}} = \nu^{-1} \sim L^{\kappa}$ and $T_{\text{kin}} = \lambda^{-2} = L^{2+\varepsilon-}$ for $\varepsilon \in (0,1)$ and $\kappa \in (0,\infty)$. Define the ratio

$$\varrho := \lim_{L\to\infty} \frac{T_{\text{kin}}}{T_{\text{for}}},$$

then there exists some $0 < \delta \ll 1$ such that, as $L \to \infty$,

$$\mathbb{E}|\widehat{u}(t,k)|^2 = n_{\text{app}}\left(\frac{t}{T_{\text{kin}}}, k\right) + \mathcal{O}_{\ell^\infty_k}\left(L^{-\delta}\,\frac{t}{T_{\text{kin}}}\right) \qquad L^{0+} \leq t \leq L^{-\varepsilon}\, T_{\text{kin}}$$

where

(i) if $\varrho = 0$, n_{app} is the 2nd order Picard iterate to the WKE

$$\begin{cases} \partial_t n = \mathcal{K}(n) \\ n(t,k)|_{t=0} = c(k)^2 \end{cases}$$

$$n_{\text{app}}(t,k) = c(k)^2 + t\,\mathcal{K}(c)(k)$$

Kinetic theory for Stochastic NLS

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where

(ii) if $\varrho = \infty$, $\quad n_{\text{app}}(t,k) = c_k^2\, e^{-2\varrho\gamma_k t} + \dfrac{b_k^2}{\gamma_k}\,(1 - e^{-2\varrho\gamma_k t})$ $\quad$ solves $\quad \begin{cases} \partial_t n(t,k) = -2\varrho\,\gamma_k\, n(t,k) + 2\varrho\, b(k)^2 \\ n(0,k) = c(k)^2 \end{cases}$

$$\gamma_k = (1+|k|^2)^r$$

Kinetic theory for Stochastic NLS

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where

(iii) If $\varrho \in (0,\infty)$, n_{app} is the 2nd order Picard iterate to the following WKE

$$\partial_t n(t,k) = \mathcal{K}(n)(t,k) - 2\varrho\,(1+|k|^2)^r\,n(t,k) + 2\varrho\,b(k)^2 \qquad\qquad n(t,k)|_{t=0} = c(k)^2$$

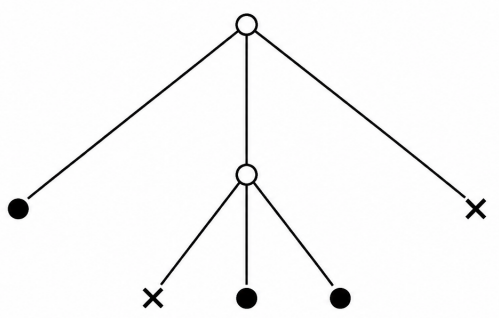
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Strategy of proof

- **Picard iteration:** $$\partial_t u = -i\Delta u + i\lambda \left(|u|^2 - \frac{2}{L^d} \int_{\mathbb{T}_L^d} |u|^2 \right) u - \nu (1-\Delta)^r u + \sqrt{\nu}\, \dot{\beta}$$

$$u = u^{(0)} + u^{(1)} + u^{(2)} + \ldots + u^{(N)} + \text{remainder}$$

$$u^{(0)}(t) = e^{-it\Delta - \nu\, t(1-\Delta)^r} u_0 + \sqrt{\nu} \int_0^t e^{-i(t-s)\Delta - (t-s)\,\nu\,(1-\Delta)^r} d\beta(s)$$



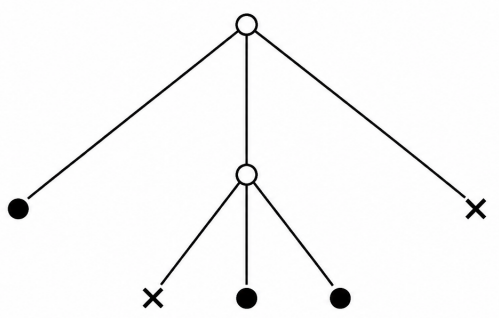
- Obtain formulas for Picard iterates, and prove (almost a.s.) bounds

Strategy of proof

- **Picard iteration:** $$\partial_t u = -i\Delta u + i\lambda \left(|u|^2 - \frac{2}{L^d} \int_{\mathbb{T}^d_L} |u|^2 \right) u - \nu (1-\Delta)^r u + \sqrt{\nu}\, \dot{\beta}$$

$$u = u^{(0)} + u^{(1)} + u^{(2)} + \ldots + u^{(N)} + \text{remainder}$$

$$u^{(0)}(t) = e^{-it\Delta - \nu\, t(1-\Delta)^r} u_0 + \sqrt{\nu} \int_0^t e^{-i(t-s)\Delta - (t-s)\,\nu\,(1-\Delta)^r} d\beta(s)$$



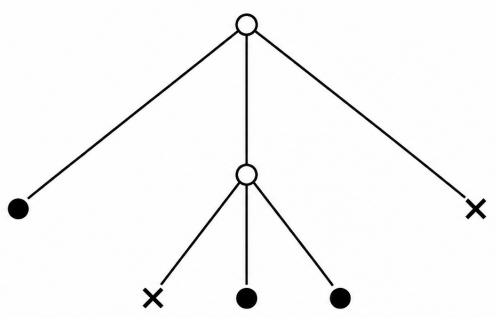
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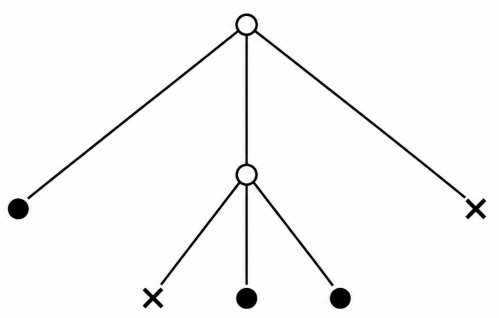
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- Match Picard iterates of WKE to those of stochastic NLS, prove the rest of the terms are small.
- Bounds on iterates & fixed point argument to control remainder.
- Obtain sharp asymptotic developments for the leading terms: derive kinetic equation.

Strategy of proof

- More complex correlations wrt to random data case: must exploit separation in frequency and wavenumber

$$u^{(0)}(t) = e^{-it\Delta - \nu\, t(1-\Delta)^r} u_0 + \sqrt{\nu} \int_0^t e^{-i(t-s)\Delta - (t-s)\,\nu\,(1-\Delta)^r} d\beta(s)$$

$$u_0(x) = L^{-d/2} \sum_{k \in \mathbb{Z}^d_L} c_k\, \eta_k\, e^{ik\cdot x} \qquad\qquad \dot\beta(t,x) = L^{-d/2} \sum_{k \in \mathbb{Z}^d_L} b_k\, e^{ik\cdot x}\, d\beta_k(t)$$

$$\mathbb{E}[\widehat{u}^{(0)}_k(\tau)\overline{\widehat{u}^{(0)}_{k'}(\tau')}] \lesssim_N c_k^2 \langle\tau\rangle^{-N} \langle\tau'\rangle^{-N} \delta_{k-k'} + b_k^2 \langle\tau-\tau'\rangle^{-N} \langle\tau\rangle^{-1} \langle\tau'\rangle^{-1} \delta_{k-k'} \qquad \forall N \in \mathbb{N}$$

Strategy of proof

- Convergence to WKE: must study delicate cancellation between delta functions in the kinetic limit

$$\mathbb{E}|u_k(t)|^2 = \mathbb{E}\left([u_k^{(0)}(t) + u_k^{(1)}(t) + u_k^{(2)}(t) + \ldots]\,[\overline{u_k^{(0)}(t)} + \overline{u_k^{(1)}(t)} + \overline{u_k^{(2)}(t)} + \ldots]\right)$$

$$= \mathbb{E}\left[|u_k^{(0)}(t)|^2 + |u_k^{(1)}(t)|^2 + 2\mathrm{Re}\,\overline{u_k^{(0)}(t)}u_k^{(2)}(t)\right] + o\left(\frac{t}{T_{\mathrm{kin}}}\right)$$

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$$\gamma_k = (1+|k|^2)^r,\ r \in (0,1]$$
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$$\mathbb{E}|u_k^{(1)}(t)|^2 = \sum_{j=1}^{3} L^{-2d} \sum_{\substack{k = k_1 - k_2 + k_3 \\ \Lambda \neq 0}} \int_0^s e^{-2\varrho\gamma_k (s-s')} \, \nu^{-1} \, h_j(\nu^{-1}\Lambda, \Gamma) \prod_{i=1}^{3} f(k_i, s') \, ds'$$

$$\Gamma = \gamma_{k_1} + \gamma_{k_2} + \gamma_{k_3} - \gamma_k$$
$$\Lambda = |k_1|^2 - |k_2|^2 + |k_3|^2 - |k|^2$$

$$h_1(x,y) = \frac{2y}{y^2 + x^2}, \qquad h_2(x,y) = -\frac{y \, \cos(\varrho(s-s')x)}{y^2 + x^2}, \qquad h_3(x,y) = \frac{x \, \sin(\varrho(s-s')x)}{y^2 + x^2}.$$

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- Exploit non-degeneracy conditions between dispersion and dissipation

Lemma: Let $\Gamma = \gamma_{k_1} + \gamma_{k_2} + \gamma_{k_3} - \gamma_k$ and $\Lambda = |k_1|^2 - |k_2|^2 + |k_3|^2 - |k|^2$ with $k = k_1 - k_2 + k_3$ then $\Lambda^2 + \Gamma \geq 1$. In particular,

$$\{\gamma_{k_1} + \gamma_{k_2} + \gamma_{k_3} - \gamma_k = 0\} \cap \{|k_1|^2 - |k_2|^2 + |k_3|^2 - |k|^2 = 0\} = \emptyset$$

Convergence to WKE

$$\mathbb{E}|u_k^{(0)}(t)|^2 = f\left(k, \frac{t}{T_{\text{kin}}}\right) \qquad f(k,s) = c_k^2\, e^{-2\varrho\gamma_k s} + \frac{b_k^2}{\gamma_k}\left(1 - e^{-2\varrho\gamma_k s}\right) \qquad \begin{array}{l} t = s\,T_{\text{kin}} \quad s \in [0,1] \\ \varrho = \nu T_{\text{kin}} \end{array}$$

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Theorem : Let $d \geq 2$, $F \in \mathcal{S}(\mathbb{R}^{3d})$ and $s > s'$. Then as $\nu \to 0$, we have that

$$\int_{\substack{k_1,k_3\in\mathbb{R}^{2d} \\ k=k_1-k_2+k_3}} \nu^{-1} h_1(\nu^{-1}\Lambda,\Gamma)\, F(k_1,k_2,k_3)\, dk_1\, dk_3 \longrightarrow \int_{\substack{k_1,k_3\in\mathbb{R}^{2d} \\ k=k_1-k_2+k_3}} 2\pi\delta(\Lambda)\, F(k_1,k_2,k_3)\, dk_1\, dk_3$$

$$\int_{\substack{k_1,k_3\in\mathbb{R}^{2d} \\ k=k_1-k_2+k_3}} \nu^{-1} h_2(\nu^{-1}\Lambda,\Gamma)\, F(k_1,k_2,k_3)\, dk_1\, dk_3 \longrightarrow -\pi \int_{\substack{k_1,k_3\in\mathbb{R}^{2d} \\ k=k_1-k_2+k_3}} e^{-\varrho(s-s')\Gamma}\, \delta(\Lambda)\, F(k_1,k_2,k_3)\, dk_1\, dk_3$$

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$$\approx 2\pi \int_{\substack{k_1,k_3\in\mathbb{R}^{2d} \\ k=k_1-k_2+k_3}} \int_0^s e^{-2\varrho\gamma_k(s-s')}\, \delta(\Lambda) \prod_{i=1}^{3} f(k_i, s')\, ds'$$

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Future perspectives

- Analysis of the damped/driven kinetic equation

$$\partial_t n(t,k) = \mathcal{K}(n,n,n)(t,k) - 2\varrho\,(1+|k|^2)^r\, n(t,k) + 2\varrho\, b(k)^2 \qquad n(t,k)|_{t=0} = c(k)^2$$

$$\mathcal{K}(n,n,n)(t,k) = \int_{\substack{(\mathbb{R}^d)^3 \\ k=k_1-k_2+k_3}} \delta\left(|k_1|^2 - |k_2|^2 + |k_3|^2 - |k|^2\right)\, [n_1 n_2 n_3 - n n_2 n_3 + n n_1 n_3 - n n_1 n_2]\, dk_1\, dk_3$$

Existence of solutions that resemble Kolmogorov-Zakharov spectra in the inertial range?

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Existence of solutions that resemble Kolmogorov-Zakharov spectra in the inertial range?

- Admissible dissipation: $(1-\Delta)^r \to (1+|k|^2)^r = \gamma_k$

Is $\{\gamma_{k_1} + \gamma_{k_2} + \gamma_{k_3} - \gamma_k = 0\} \cap \{|k_1|^2 - |k_2|^2 + |k_3|^2 - |k|^2 = 0\} = \emptyset$ necessary?

No, our results hold for $\gamma_k = |k|^2$ which violates the condition.

A quantitative version of this condition might be sharp?

Thank you !