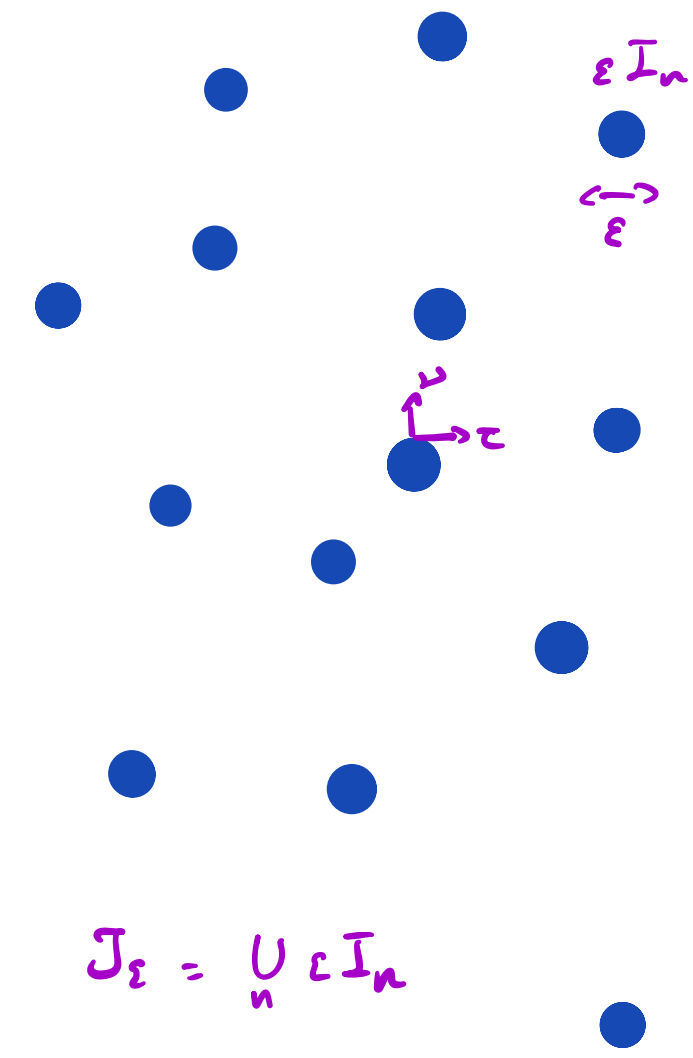
Homogenization of the 2D Euler system in porous media

M. Duerinckx & A. Gloria

$0 < \varepsilon << 1$



$$\begin{cases} \partial_t u_\varepsilon + u_\varepsilon \cdot \nabla u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon & \mathbb{R}^2 \setminus T_\varepsilon \\ \operatorname{div} u_\varepsilon = 0 & \mathbb{R}^2 \setminus T_\varepsilon \end{cases}$$

$$\begin{cases} u_\varepsilon \cdot \nu = 0 & \text{on } \partial T_\varepsilon \\ \int_{\varepsilon \partial I_n} u_\varepsilon \cdot \tau = 0 & \text{for all } n \end{cases}$$

$$u_\varepsilon|_{t=0} = u_\varepsilon^0$$

What happens when $\varepsilon \searrow 0$?

① weakly nonlinear $\partial_t u_\varepsilon + \varepsilon\, u_\varepsilon \cdot \nabla u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon$

→ well-understood, multiscale limit [Lions - Masmoudi '05] [Mihelič - Paoli '99]

② nonlinear, obstacles $\mathcal{I}_\varepsilon \subset \mathcal{D}$ bounded domain

- if $|\mathcal{I}_\varepsilon| \to |\mathcal{D}|$ then Euler in $\mathbb{R}^2 \setminus \overline{\mathcal{D}}$
- if $|\mathcal{I}_\varepsilon| \to 0$ then Euler in $\mathbb{R}^2$

[Bonnaillie-Noël, Lacave - Masmoudi '15 & '16]

③ nonlinear, obstacles $\mathcal{I}_\varepsilon \subset \mathcal{D}$ bounded domain

- if $|\mathcal{I}_\varepsilon| \to \lambda\, |\mathcal{D}|$ with $0 < \lambda < 1$
- if $\operatorname{supp}(\operatorname{curl} u_\varepsilon) \subset \mathbb{R}^2 \setminus \overline{\mathcal{D}}$

→ then Euler in $\mathbb{R}^2 \setminus \overline{\mathcal{D}}$ to leading order

[Hillairet - Lacave WW '22]

Remaining question : $\begin{cases} |\mathcal{I}_\varepsilon \cap \mathcal{D}| \to \lambda\, |\mathcal{D}|, & 0 < \lambda < 1 \\ \operatorname{supp}(\operatorname{curl} u_\varepsilon) \cap \mathcal{D} \neq \emptyset \end{cases}$

[general case]

VORTICITY FORMULATION

transport equation

$$\partial_t \omega_\varepsilon + \operatorname{div}(u_\varepsilon \omega_\varepsilon) = g_\varepsilon := \operatorname{curl}(f_\varepsilon) \qquad \mathbb{R}^2 \setminus T_\varepsilon$$

div - curl system

$$\begin{cases} \operatorname{div}(u_\varepsilon) = 0 & \mathbb{R}^2 \setminus T_\varepsilon \\ \operatorname{curl}(u_\varepsilon) = \omega_\varepsilon & \mathbb{R}^2 \setminus T_\varepsilon \\ u_\varepsilon \cdot \nu = 0 & \partial T_\varepsilon \\ \int_{\varepsilon \partial I_n} u_\varepsilon \cdot \tau = 0 & \forall n \end{cases}$$

$$\omega_\varepsilon|_{t=0} = \omega_\varepsilon^0 := \operatorname{curl}(u_\varepsilon^0)$$

Two tasks :

- homogenize the div - curl system
- homogenize the transport equation

↳ subtle part

For simplicity : T_ε ε-periodic

(later on, ε-rescaling of stationary & ergodic T)

MAIN RESULT (Duerinckx-G. '24)

Assume $\begin{cases} \operatorname{curl}(u_\varepsilon^0)\,\mathbb{1}_{\mathbb{R}^2\setminus\mathfrak{I}_\varepsilon} \overset{*}{\rightharpoonup} \omega^0 & \text{in } L^1\cap L^\infty(\mathbb{R}^2) \\ f_\varepsilon\,\mathbb{1}_{\mathbb{R}^2\setminus\mathfrak{I}_\varepsilon} \overset{*}{\rightharpoonup} f & \text{in } L^\infty_{loc}(\mathbb{R}_+;L^2(\mathbb{R}^2)) \\ \operatorname{curl}(f_\varepsilon)\,\mathbb{1}_{\mathbb{R}^2\setminus\mathfrak{I}_\varepsilon} \overset{*}{\rightharpoonup} g & \text{in } L^\infty_{loc}(\mathbb{R}_+;L^1\cap L^\infty(\mathbb{R}^2)) \end{cases}$

Along a subsequence, $\begin{cases} \operatorname{curl}(u_\varepsilon)\,\mathbb{1}_{\mathbb{R}^2\setminus\mathfrak{I}_\varepsilon} \overset{*}{\rightharpoonup} \omega & \text{in } L^\infty_{loc}(\mathbb{R}_+;L^1\cap L^\infty(\mathbb{R}^2)) \\ u_\varepsilon\,\mathbb{1}_{\mathbb{R}^2\setminus\mathfrak{I}_\varepsilon} \overset{*}{\rightharpoonup} u & \text{in } L^\infty_{loc}(\mathbb{R}_+;L^2(\mathbb{R}^2)) \end{cases}$

If the cluster point (ω,u) is nowhere one-dimensional with rational direction then it is the unique solution of

$$\begin{cases} \partial_t\omega + \frac{1}{1-\lambda}\operatorname{div}(\omega u) = g \\ \operatorname{div}(u) = 0 \\ \operatorname{curl}(\bar m\, u) = \omega \\ \omega_{|t=0} = \omega^0 \end{cases} \qquad \text{in } \mathbb{R}^2$$

with $\lambda = |\mathfrak{I}\cap Q|$, $\bar m = J^T\bar a\, J \in \mathbb{R}^{2\times 2}$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$e_j\cdot\bar a e_i = \int_Q (\nabla\varphi_j + e_j)\cdot(\nabla\varphi_i + e_i)$

$$\begin{cases} -\Delta\varphi_i = 0 & \text{in } Q\setminus(\mathfrak{I}\cap Q) \\ \nabla\varphi_i + e_i = 0 & \text{in } \cup I_n \\ \int_{\partial I_n}\partial_\nu\varphi_i = 0 & \forall\, n \end{cases}$$

COMMENTS

① Biot-Savart law : $u = \nabla^\perp \Delta^{-1} \omega$ Euler

$u = \nabla^\perp (\operatorname{div} \bar{a} \nabla)^{-1} \omega$ here

② Div-curl problem is elliptic and easy to homogenize

but still : $-\operatorname{div} a \nabla$ with $a = \infty$ in inclusions

③ Homogenization of transport equations usually subtle

linear : $\partial_t \omega_\varepsilon + b(\frac{x}{\varepsilon}) \cdot \nabla \omega_\varepsilon = g$ + IC

→ in good cases (uniqueness of invariant measure)

$\partial_t \omega + \langle b \rangle \cdot \nabla \omega = g$ (Weinan E)

→ examples for which no homogenization

(Jabin - Tzavaras)

but seen using kinetic formulation

Div-Curl Problem

$$\begin{cases} \operatorname{div}(u_\varepsilon) = 0 & \mathbb{R}^2 \setminus T_\varepsilon, \\ \operatorname{curl}(u_\varepsilon) = \omega_\varepsilon & \mathbb{R}^2 \setminus T_\varepsilon. \end{cases}$$

Note $\omega_\varepsilon = \operatorname{curl}(f_\varepsilon)$

[$\omega_\varepsilon \mathbb{1}_{\mathbb{R}^2 \setminus T_\varepsilon} \in H^{-1}(\mathbb{R}^2)$ because $f_\varepsilon \cdot \tau = 0$ on ∂I_ε]

Equivalent to $u_\varepsilon = \nabla^\perp s_\varepsilon$, where

$$\begin{cases} \Delta s_\varepsilon = \omega_\varepsilon & \text{in } \mathbb{R}^2 \setminus T_\varepsilon \\ \nabla s_\varepsilon = 0 & \text{in } T_\varepsilon \\ \int_{\varepsilon \partial I_n} \partial_\nu s_\varepsilon = 0 & \forall n \end{cases}$$

(i.e. rigid inclusions)

Since $\omega_\varepsilon \mathbb{1}_{\mathbb{R}^2 \setminus T_\varepsilon} \in H^{-1}(\mathbb{R}^2)$, ∇s_ε uniquely defined in $L^2(\mathbb{R}^2)^2$

Homogenization: $\nabla s_\varepsilon \rightharpoonup \nabla s$ in $L^2(\mathbb{R}^2)^2$, $\operatorname{div}(\bar{a} \nabla s) = \omega$ in $\mathbb{R}^2$.

$e_j \cdot \bar{a} e_i = \int_Q (\nabla \varphi_j^0 + e_j) \cdot (\nabla \varphi_i + e_i)$

$$\begin{cases} -\Delta \varphi_i = 0 & \text{in } Q \setminus (T \cap Q) \\ \nabla \varphi_i + e_i = 0 & \text{in } \cup I_n \\ \int_{\partial I_n} \partial_\nu \varphi_i = 0 & \forall n \end{cases}$$

Div-Curl Problem

Weak limit ∇s of ∇s_ε is not enough for the transport equation

Need corrector result : $\Phi = \varphi_1 e_1 + \varphi_2 e_2$

$$\nabla s_\varepsilon - [Id + \nabla \Phi(\cdot/\varepsilon)]\nabla s \longrightarrow 0 \quad \text{in } L^2(\mathbb{R}^2)^2$$

Not that simple : s_ε depends also on time because w_ε does !

What we need is

$$\boxed{\nabla s_\varepsilon - [Id + \nabla \Phi(\cdot/\varepsilon)]\nabla s \longrightarrow 0 \text{ in } L^\infty_{loc}(\mathbb{R}_+; L^2(\mathbb{R}^2)^2)}$$

[combination of arguments à la Aubin-Lions for time
+ "standard" corrector arguments for bounded domains
+ technical ingredients for whole space]

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div - curl system

$$\begin{cases} \operatorname{div}(u_\varepsilon) = 0 & \mathbb{R}^2 \setminus T_\varepsilon \\ \operatorname{curl}(u_\varepsilon) = \omega_\varepsilon & \mathbb{R}^2 \setminus T_\varepsilon \\ u_\varepsilon \cdot \nu = 0 & \partial T_\varepsilon \\ \int_{\varepsilon \partial I_n} u_\varepsilon \cdot \tau = 0 & \forall n \end{cases}$$

$$\omega_\varepsilon|_{t=0} = \omega_\varepsilon^0 := \operatorname{curl}(u_\varepsilon^0)$$

Two tasks : + homogenize the div - curl system

$$\boxed{\begin{aligned} \operatorname{div}(u) &= 0 \\ \operatorname{curl}(\bar{m}\, u) &= \omega \end{aligned}} \quad \checkmark$$

$$u_\varepsilon \mathbb{1}_{\mathbb{R}^2 \setminus T_\varepsilon} + \underbrace{(e_i + \nabla \varphi_i)^\perp \left(\tfrac{\cdot}{\varepsilon}\right) u_i^\perp}_{\text{multiscale velocity for ...}} \to 0 \text{ in } L^\infty_{loc}(\mathbb{R}_+, L^2(\mathbb{R}^2)^2) \quad \checkmark$$

+ homogenization of the transport equation

TRANSPORT EQUATION

$\tilde{u}_\varepsilon = u_\varepsilon \mathbb{1}_{\mathbb{R}^2 \setminus \mathcal{K}}$, same for $\tilde{\omega}_\varepsilon$ and $\tilde{f}_\varepsilon$

$$\boxed{\partial_t \tilde{\omega}_\varepsilon + \operatorname{div}(\tilde{u}_\varepsilon \tilde{\omega}_\varepsilon) = \tilde{f}_\varepsilon \quad \text{in } \mathbb{R}^2}$$

A priori estimate : $\sup_\varepsilon \| \tilde{\omega}_\varepsilon \|_{L^2(0,T; L^2(\mathbb{R}^2))} < \infty$, $\forall T$

By L^2-compactness : $\exists\, \bar{W} \in L^2_{loc}(\mathbb{R}_+ \times \mathbb{R}^2, L^2(Q, \mathbb{R}^2))$

s.t. $\forall\, \psi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$, $\Theta \in C^\infty_{per}(Q)$

$$\int_{\mathbb{R}_+ \times \mathbb{R}^2} \tilde{\omega}_\varepsilon(t,x) \psi(t,x) \Theta\left(\frac{x}{\varepsilon}\right) \xrightarrow[\varepsilon \to 0]{\text{extraction}} \int_Q \int_{\mathbb{R}_+ \times \mathbb{R}^2} \bar{W}(t,x,y) \psi(t,x) \Theta(y)\, dt\, dx\, dy$$

① weak limit ω of $\tilde{\omega}_\varepsilon$ satisfies $\omega(t,x) = \int_Q \bar{W}(t,x,y)\, dy$

② Can we characterize $\bar{W}$?
Is it unique ?

TRANSPORT EQUATION

Equations satisfied by $\bar{w}$ (choose ψ, Θ properly use $\tilde{\pi}_\varepsilon + (e_i + \nabla p_i)^\perp(1/\varepsilon)\, u_i^\perp \to 0$)

$$\begin{cases} \bar{w}(t,x,y) = 0 \quad \text{for } y \in \mathcal{I} \cap Q, \text{ a.e } t,x \\ \operatorname{div}_y [\bar{w}(t,x,y)\, R_{t,x}(y)] = 0 \quad \text{in } Q \setminus \mathcal{I}, \text{ a.e } t,x \\ \qquad R_{t,x}(y) = (e_i + \nabla p_i)^\perp(y)\, u_i^\perp(t,x) \\ \partial_t w(t,x) + \operatorname{div}_x \left(\int_Q \bar{w}(t,x,y)\, R_{t,x}(y)\, dy \right) = g \quad \text{in } \mathbb{R}^2 \\ w|_{t=0} = w^0 \end{cases}$$

a priori not closed !

need at least $\int_Q \bar{w}(t,x,y)\, R_{t,x}(y)\, dy$ to be independent of the solution $\bar{w}$ (if more than one)

Herman rotation set

TRANSPORT EQUATION AND DYNAMICAL SYSTEMS

KEY : $\operatorname{div}_y [W(t,x,y)\, R_{t,x}(y)] = 0$ in $\mathcal{D}'$, a.e t,x

$R_{t,x}(y) = (e_i + \nabla p_i)^{\perp}(y)\, u_i^{\perp}(t,x)$

We say: $W(t,x,\cdot)$ is an invariant measure for $R_{t,x}(\cdot)$

(t,x) fixed parameter

ODE : $\begin{cases} \dot{y}(s;\bar{y}) = R(y(s;\bar{y})) \text{ in } \mathbb{R}^2 \setminus \Gamma \\ y(0;\bar{y}) = \bar{y} \in \mathbb{R}^2 \setminus \Gamma \end{cases}$ (well-posed)

For all $\bar{Y} \subset \mathbb{R}^2 \setminus \Gamma$ bounded

$$\int_{\bar{Y}} W(y(s;\bar{y}))\, d\bar{y} = \int_{\bar{Y}} \bar{W}(\bar{y})\, d\bar{y} \quad \forall s \geq 0$$

$$\frac{d}{ds} \int_{\bar{Y}} W(y(s;\bar{y}))\, d\bar{y} = \int_{\bar{Y}} \nabla W(y(s;\bar{y})) \cdot \dot{y}(s;\bar{y})\, d\bar{y}$$

$$= \int_{\bar{Y}} \nabla W(y(s;\bar{y})) \cdot R(y(s;\bar{y}))\, d\bar{y}$$

$\operatorname{div} R \equiv 0$ ⟹

$$= \int_{\bar{Y}} \operatorname{div}\left[W(y(s;\bar{y}))\, R(y(s;\bar{y})) \right] d\bar{y} \overset{\text{KEY eq.}}{=} 0$$

A RESULT BY ARNOLD (PERIODIC)

On $\mathbb{R}^2 \setminus \mathcal{I}$, the dynamical system

$$\begin{cases} \dot{y}(s;\bar{y}) = R(y(s;\bar{y})) \\ y(0;\bar{y}) = \bar{y} \end{cases}$$

with $R(z) = e \cdot z + \xi(z)$ (ξ: $\mathbb{Q}$-periodic)

and $e = (e_1, e_2) \in \mathbb{R}^2$,

admits $W \equiv ct$ as unique invariant measure

if (A1) R does not vanish on $\mathbb{R}^2 \setminus \mathcal{I}$

(A2) $e_1/e_2 \notin \mathbb{Q}$

Question: what can we say about

$$R_{t,x}(y) = (e_i + \nabla p_i)^{\perp}(y)\, u_i^{\perp}(t,x)$$

BACK TO TRANSPORT EQUATION

Assume (A1) & (A2), by Arnold $W(t,x,y) = \bar{W}(t,x)$ $y \in Q_1^\lambda$

Then

- $\omega(t,x) = \int_Q W(t,x,y)\,dy = \bar{W}(t,x)\,|Q_1^\lambda| = \bar{W}(t,x)(1-\lambda)$

  $\Rightarrow \quad \bar{W}(t,x) = \frac{1}{1-\lambda}\,\omega(t,x)$

- Macroscopic transport equation : drift term

$$\int_Q W(t,x,y)\,(e_i + \nabla\varphi_i)^{-1}(y)\,u_i^\varepsilon(t,x)\,dy$$

$$= \bar{W}(t,x)\,u_i^{-1}(t,x) \underbrace{\int_{Q_1^\lambda} (e_i + \nabla\varphi_i)^{-1}(y)\,dy}_{= e_i}$$

$$= \frac{1}{1-\lambda}\,\omega(t,x)\,u(t,x)$$

$\rightsquigarrow$

$$\boxed{\partial_t \omega + \frac{1}{1-\lambda}\,\mathrm{div}(u\omega) = g \quad \text{in } \mathbb{R}^2 \qquad \omega|_{t=0} = \omega^0}$$

CONCLUSION

Under conditions (A1) $e + \nabla\varphi_i$ does not vanish on $\mathcal{Q}\setminus\mathcal{T}$

(A2) $e = u(t,x)$ irrational a.e.

+ assumptions on f_ε, u_ε° (and their curl)

we obtain $(u_\varepsilon \mathbb{1}_{\mathcal{M}^2\setminus\mathcal{T}_\varepsilon}, (\mathrm{curl}\, u_\varepsilon)\mathbb{1}_{\mathcal{M}^2\setminus\mathcal{T}_\varepsilon}) \rightharpoonup (u, \omega)$

where (u,ω) solves

$$\partial_t \omega + \frac{1}{1-\lambda}\,\mathrm{div}(u\omega) = g \quad \text{in } \mathcal{M}^2$$

$$\omega_{|t=0} = \omega^\circ$$

$$\mathrm{div}(u) = 0$$

$$\mathrm{curl}(\bar{m}\, u) = \omega$$

Condition (A2) is a "genericity" assumption on weak limit of $u_\varepsilon \mathbb{1}_{\mathcal{M}^2\setminus\mathcal{T}_\varepsilon}$

(A1) is true!

CONDITION (A1)

sufficient for us

Statement :

$\forall\, e \in \mathbb{R}^2$

$$\begin{cases} \Delta\varphi = 0 & \text{on } Q\setminus\overline{J} \\ \nabla\varphi \cdot e = 0 & \text{on } J \\ \int_{\partial I_n} \partial_\nu \varphi = 0 & \forall n \end{cases} \Rightarrow \nabla\varphi(y)\cdot e \neq 0 \quad \text{for all } y \in Q\setminus\overline{J}$$

Reformulation

$$\begin{cases} \Delta\Phi = 0 & \text{on } Q\setminus J \\ \nabla\Phi + Id = 0 & \text{on } J \\ \int_{\partial I_n} \partial_\nu \Phi = 0 & \forall n \end{cases} \Rightarrow \det(\nabla\Phi + Id) \neq 0 \quad \text{for all } y \in Q\setminus\overline{J}$$

Variant of the Radó - Kneser - Choquet theorem

$$\begin{cases} \Delta\Phi = 0 & \text{in } B \\ \Phi = \psi & \text{on } \partial B \end{cases}$$

where $\psi : \partial B \to \mathbb{R}^2$ is a homeomorphism

then Φ is a diffeomorphism from B to $\Phi(B)$.

MORE ON HARMONIC COORDINATES

Theorem (Alessandrini - Nesi '01 '18)

Let a be a Hölder-continuous periodic uniformly elliptic symmetric matrix field, and let $\Phi \in H^1_{per}(Q)$ be the unique solution of $\operatorname{div} a(\nabla\Phi + Id) = 0$.

Then $x \mapsto x + \Phi(x)$ is a diffeomorphism on Q.

$\Rightarrow \det(\nabla\Phi + Id)(x) \neq 0$ for all $x \in Q$.

(motivated by G-closure in dimension 2)

Corollary (Duerinckx - G. '24)

- same conclusion for a Hölder continuous stationary and ergodic (using qualitative large-scale regularity)
- similar conclusion for still periodic coefficients:

  If $\Delta\Phi = 0$ in $Q\setminus T$, $\nabla\Phi + Id = 0$ on T, $\int_{\partial T_n} \partial_\nu \Phi = 0$ $\forall n$

  then Φ is a diffeomorphism on $Q\setminus T$.
- same for still stationary and ergodic coefficients

SKETCH OF PROOF

Periodic setting, with one inclusion $I = B_{1/2}(0)$.

Approximation $a_N(x) = Id + N\eta_N(x)$

$\eta_N = 1$ in $B_{1/2 - 1/N}(0)$

$\eta_N = 0$ in $Q \setminus B_{1/2}(0)$

$0 \le \eta_N \le 1$, $\eta_N \in C^1(Q)$

$a_N \in C^1(Q)$, $a_N \le a_{N+1}$

$a_N(x) = Id$ if $x \in Q \setminus I$

$a_N(x) \xrightarrow[N \to \infty]{} +\infty$ if $x \in I$

By monotonicity

$$J_N : H^1_{per}(Q) \to \mathbb{R}_+$$
$$v \mapsto \frac{1}{2} \int_Q \nabla v \cdot a_N \nabla v$$

Γ(w H^1)-converges to

$$J : H^1_{per}(Q) \to \mathbb{R}_+ \cup \{+\infty\}$$
$$v \mapsto \begin{cases} \frac{1}{2} \int_Q |\nabla v|^2 & \text{if } \nabla v|_I \equiv 0 \\ +\infty & \text{else} \end{cases}$$

Call Φ_N the given approximation of Φ

we have $D\Phi_N \to D\Phi$ in $L^2(\Omega \setminus I)$

① By "reconstructing a square": $D\Phi_N \to D\Phi$ in $L^2(\Omega \setminus I)$

② By "harmonicity": $D\Phi_N \to D\Phi$ in $C^0(\overline{\Omega \setminus I})$

Up to now: (1) $\begin{cases} \det(Id + D\Phi_N) \neq 0 \quad \forall x \in \Omega \\ D\Phi_N \to D\Phi \text{ in } C^0(\Omega \setminus I) \end{cases}$

Is it enough to conclude that $\det D\Phi(x) \neq 0 \quad \forall x \in \Omega \setminus I$?

Indeed: the number of critical points of a harmonic function in dimension 2 is given by a contour integral involving its gradient (complex analysis)

Hence, (1) $\Rightarrow$ $\det(Id + D\Phi)(x) \neq 0 \quad \forall x \in \Omega \setminus I$

OPEN PROBLEM : THE RANDOM SETTING

Everything is fine in the analysis except the uniqueness of the invariant measure

Question : Let $e \in S^1$, and $\mu \in L^2_{loc}(\mathbb{R}^2, L^2(\Omega))$ be a stationary random field that satisfies almost surely in the weak sense

$$\operatorname{div}\left(\mu (e + \nabla p_e)^\perp\right) = 0 \quad \text{in } \mathbb{R}^2.$$

Is μ almost surely constant in $\mathbb{R}^2 \setminus \mathcal{J}$?

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Is μ almost surely constant in $\mathbb{R}^2 \setminus \mathcal{J}$?

Positive answer : If $\underset{\sim}{\cup} I_n$ is stationary and strongly mixing, then μ is almost surely constant in $\mathbb{R}^2 \setminus \mathcal{J}$ if $\sup \mathbb{E}\left[|\varphi(x)|^2\right] < \infty$.

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Is μ almost surely constant in $\mathbb{R}^2 \setminus \mathcal{J}$?

Positive answer : If $\underset{\sim}{\nabla} I_n$ is stationary and strongly mixing, then μ is almost surely constant in $\mathbb{R}^2 \setminus \mathcal{J}$ if $\sup_x \mathbb{E}\left[|\varphi(x)|^2\right] < \infty$.

Problem : For $d \geq 3$, if $\underset{\sim}{\nabla} I_n$ is enough mixing, then $\sup_x \mathbb{E}\left[|\varphi(x)|^2\right] < \infty$

But for $d = 2$, $\mathbb{E}\left[|\varphi(x)|^2\right] \gtrsim \log(2 + |x|)$

Proof if φ stationary itself and $\mathbb{E}[|\varphi|^2] < \infty$

① Transport equation : reduction of dimension

$$\operatorname{div}\left(\mu (e + \nabla\varphi_e)^\perp\right) = 0$$

$\Rightarrow \mu$ is constant along characteristics of $(e + \nabla\varphi_e)^\perp$

$\Rightarrow \exists\ \mu_0 : \mathbb{R} \times \Omega \to \mathbb{R}$ such that

$$\mu(x) = \mu_0\left(\underbrace{x}_{\in \mathbb{R}^2} \cdot e + \underbrace{\varphi_e(x)}_{\in \mathbb{R}}\right) \quad \forall\, x \in \mathbb{R}^2 \setminus \mathcal{T}$$

② Harmonic coordinates are injective on $\mathbb{R}^2 \setminus \mathcal{T}$

$\mathbb{R}^2 \setminus \mathcal{T} \to \mathbb{R}$, $x \mapsto x \cdot e + \varphi_e(x)$ is almost surely injective

③ First use of stationarity and first invariance

Call T_z the group action on Ω

By def : $(T_z \mu_0)\left[x \cdot e + T_z \varphi_e(x)\right]$

$= T_z \mu\,(x)$

$= \mu(x + z)$ [stationarity]

$= \mu_0\left[x \cdot e + z \cdot e + \varphi_e(x + z)\right]$

By stationarity $(T_t\varphi_e)(x) - (T_t\varphi_e)(0) = \varphi_e(x+t) - \varphi_e(t)$

and thus

$$(T_t\mu_0)(x\cdot e + (T_t\varphi_e)(x)) = \mu_0\big(\underbrace{x\cdot e + (T_t\varphi_e)(x)}_{\text{a.s. surjective}} + t\cdot e - (T_t\varphi_0)(0)\big)$$

$$\Rightarrow (T_t\mu_0)(0) = \mu_0(t\cdot e) \quad \forall t \in \mathbb{R}^2$$

④ Directional ergodicity. Set $\bar{\mu}_0 := \mathbb{E}[\mu_0]$ (a function)

By ③, μ_0 is invariant under $\{T_{se^\perp}\}_{s\in\mathbb{R}}$

Strong mixing $\Rightarrow$ directional ergodicity

Hence: $\mu_0(0) = \bar{\mu}_0(0) := \mathbb{E}[\mu_0(0)]$ a.s.

Since 0 plays no role: $\forall r\in\mathbb{R}$, $\mu_0(r) = \bar{\mu}_0(r) = \mathbb{E}[\mu_0(r)]$ a.s

⑤ Second use of stationarity:

$$\mu(x) \overset{\text{def}}{=} \bar{\mu}_0(x\cdot e + \varphi_e(x))$$

$$\overset{\text{stat}}{=} (T_{-t}\mu)(t+x)$$

$$\overset{\text{def}\,④}{=} \bar{\mu}_0(t\cdot e + x\cdot e + (T_{-t}\varphi_e)(x+t))$$

$$\overset{\text{stat.}\,④}{=} \bar{\mu}_0(t\cdot e + x\cdot e + \varphi_e(x)) \quad \forall t \Rightarrow \bar{\mu}_0 \text{ is constant.}$$