Core size of large partitions

under the Plancherel measure

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1 Plancherel measure

2 Core of a partition
- Descent set
- Rim hooks
- Core

3 Core asymptotics under the Plancherel measure

Partitions

Let $n \in \mathbb{Z}_{\geq 0}$.

Definition

A partition of (size) n is a non increasing sequence of positive integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_h > 0)$ with sum n.

Example

The partitions of 5 are (5), $(4,1)$, $(3,2)$, $(3,1,1)$, $(2,2,1)$, $(2,1,1,1)$, $(1,1,1,1,1)$.

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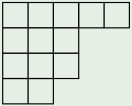
One can picture a partition with its Young diagram.

Example

The Young diagram of the partition $(5, 3, 3, 2)$ is  .

Plancherel measure

Let λ be a partition of n. A standard tableau of shape λ is a labelling of the boxes of the Young diagram of λ with the integers $1, \ldots, n$ such that the rows (resp. columns) are increasing from left to right (resp. top to bottom).

Example

The tableau $\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 6 & 7 \\ \hline 4 \\ \cline{1-1}\end{array}$ is standard with shape $(3, 3, 1)$.

We denote by $\mathrm{std}(\lambda)$ the number of standard tableaux with shape λ.

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Example

The tableau

1	2	5
3	6	7
4		

is standard with shape $(3, 3, 1)$.

We denote by $\mathrm{std}(\lambda)$ the number of standard tableaux with shape λ.

Proposition

$$n! = \sum_{\lambda \text{ partition of } n} \mathrm{std}(\lambda)^2$$

The Plancherel measure on the set of partitions of n is defined by:

$$\mathrm{Pl}_n(\lambda) := \frac{\mathrm{std}(\lambda)^2}{n!}.$$

Russian convention

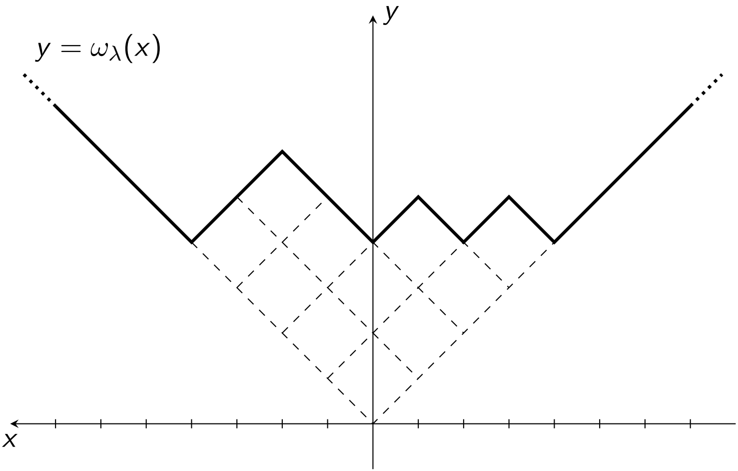

Figure: Russian convention for the partition $(4, 4, 2, 1)$.

Limit shape theorem

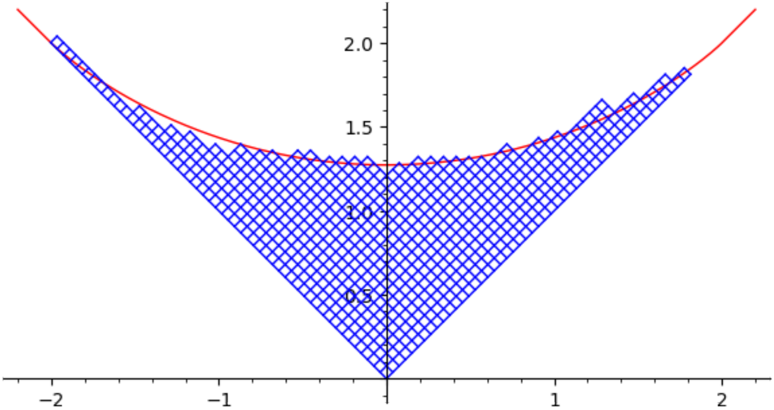

Figure: A partition of $n = 700$ and the limit shape (Kerov–Vershik, Logan–Shepp, 1977).

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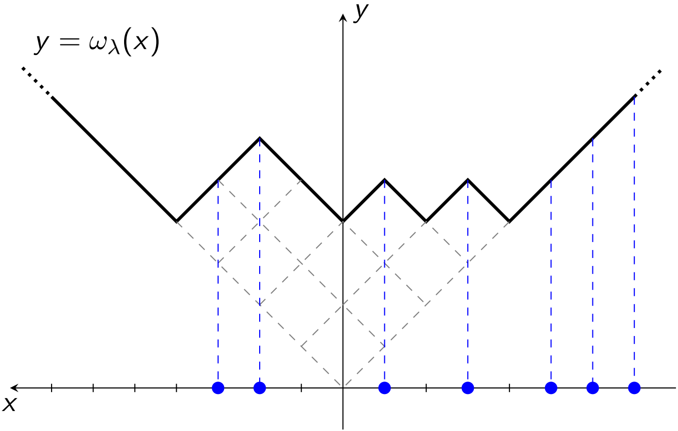
Descent set

Definition

The descent set associated with a partition $\lambda = (\lambda_i)_{i \geq 1}$ is:

$$\mathcal{D}(\lambda) := \{\lambda_i - i : i \geq 1\} \subseteq \mathbb{Z}.$$

For instance, $\mathcal{D}(4,4,2,1) = \{3, 2, -1, -3, -5, -6, -7, \dots\}$.



A determinantal process

The discrete Bessel kernel is defined for $x, y \in \mathbb{R}$ by:

$$\mathcal{J}^n(x,y) := \sqrt{n}\frac{J_x J_{y+1} - J_{x+1} J_y}{x-y}(2\sqrt{n}),$$

where J_x is the Bessel function of the first kind of order x.

Theorem (Borodin-Okounkov-Olshanski 2000)

Let $x_1, \ldots, x_s \in \mathbb{Z}$ be distinct. Under the (Poissonised) Plancherel measure pl_n we have:

$$\mathrm{pl}_n(x_1, \ldots, x_s \in \mathcal{D}(\lambda)) = \det\left[\mathcal{J}^n(x_a, x_b)\right]_{1 \le a,b \le s}. \qquad (\diamondsuit)$$

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Hooks and their rims

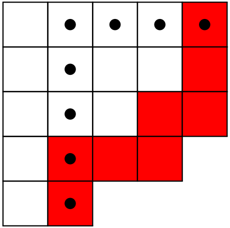

Figure: A hook (with •) and its corresponding rim hook (in red) for $\lambda = (5, 5, 5, 4, 2)$.

Link between rim hooks and beads

Proposition

Let λ, μ be two partitions. The Young diagram of μ is obtained by removing a rim hook of size e in the Young diagram of λ if and only if:

$$\mathcal{D}(\mu) = (\mathcal{D}(\lambda) \setminus \{b\}) \cup \{b - e\},$$

for a certain $b \in \mathcal{D}(\lambda)$ with $b - e \notin \mathcal{D}(\lambda)$.

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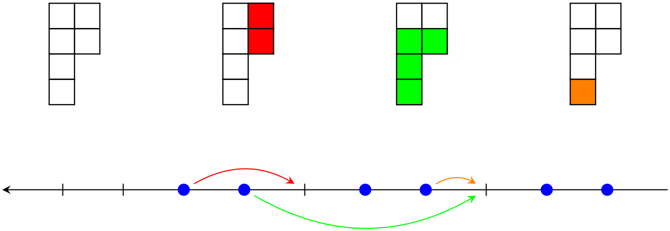
With $\lambda := (2, 2, 1, 1)$ one has $\mathcal{D}(\lambda) = (1, 0, -2, -3, -5, -6, \dots)$ and:



1. Plancherel measure
2. Core of a partition
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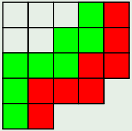
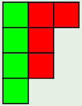
Core of a partition

Let $e \geq 1$.

Definition (Core)

The *e*-core of a partition is the partition that we obtain after we have removed all the possible rim hooks of size e of the Young diagram.

Example

- The 8-core of $(5,5,5,4,2)$ is $(3,2)$:  .
- The 4-core of $(3,2,2,1)$ is empty: 

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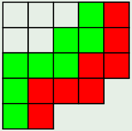
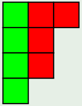
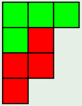
- The 8-core of $(5,5,5,4,2)$ is $(3,2)$:  .
- The 4-core of $(3,2,2,1)$ is empty:  or  .

Is the e-core well-defined?

Proposition

The e-core of a partition λ is obtained by sliding all the beads in $\mathcal{D}(\lambda)$ as far as possible to the right in their class of congruence modulo e.

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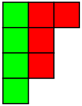
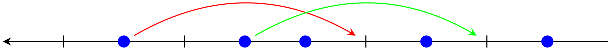
With $\lambda = (3, 2, 2, 1)$ and $e = 4$ as before:

- the order  corresponds to:



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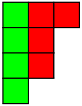
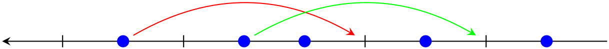
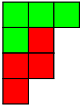
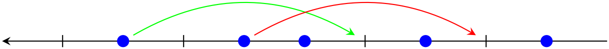
- the order  corresponds to:



- the order  corresponds to:



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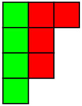
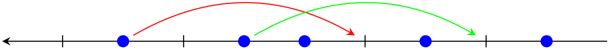
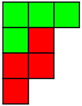
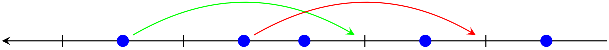
- the order  corresponds to:



- the order  corresponds to:



the 4-core being:

Partitions with the same core

Theorem ("Nakayama's Conjecture", Brauer–Robinson 1947)

Two partitions belong to the same p-block of $\mathfrak{S}_n$ if and only if they have the same p-core.

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Proposition (James–Kerber)

Two partitions have the same e-core if and only if they have the same multiset of e-residues.

Example

- The partition $(3, 2, 2, 1)$ has empty 4-core and its multiset of 4-residues is given by

0	1	2
3	0	
2	3	
1		

.

- The partition $(4, 4)$ has empty 4-core and its multiset of 4-residues is given by

0	1	2	3
3	2	1	0

.

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2 Core of a partition
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3 Core asymptotics under the Plancherel measure

How does a core of a large partition look like?

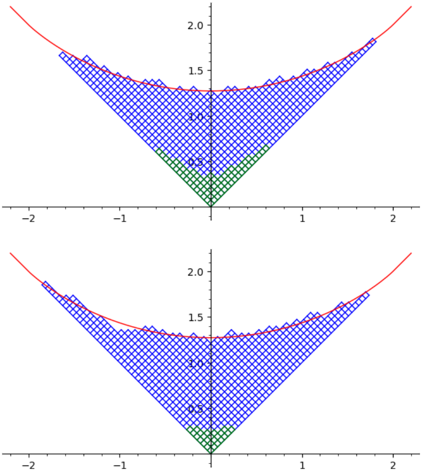

Figure: Some 5-cores (in green) for $n = 700$.

Computing the core size

For $i \in \mathbb{Z}/e\mathbb{Z}$, the number of boxes of residue i in the Young diagram of a partition λ is:

$$c_i(\lambda) = \frac{1}{2} \sum_{k \in \mathbb{Z}} \omega_\lambda(i + ke) - |i + ke| \in \mathbb{N}.$$

Define:

$$x_i(\lambda) := c_i(\lambda) - c_{i+1}(\lambda) \in \mathbb{Z}.$$

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Proposition (Garvan-Kim-Stanton 1990, Fayers 2006)

The size $\ell_e(\lambda)$ of the e-core of λ is given by:

$$\ell_e(\lambda) = \frac{e}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} x_i(\lambda)^2 + \sum_{i=0}^{e-1} i x_i(\lambda). \qquad (\Diamond)$$

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Remark (Back to partitions with the same core)

One can show that $x_i(\lambda) = x_i(\mu)$ for all $i \in \mathbb{Z}/e\mathbb{Z}$ if and only if λ and μ share the same e-core.

Central limit theorem

Proposition (R. 21)

For all $i \in \{0, \dots, e-1\}$ *one has:*

$$x_i(\lambda) = \#(e\mathbb{Z}_{\geq -n^2} + i) \cap \mathcal{D}(\lambda) - n^2 + R(\lambda), \qquad (\diamondsuit)$$

where $R(\lambda) \xrightarrow[n \to +\infty]{\mathrm{L}^2} 0$.

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We denote by $\mathbb{E}_n, \mathrm{Var}_\mathsf{n}$ the expectation and variance under pl_n.

Theorem (Costin–Lebowitz 1995)

Define $\#_i := \#(e\mathbb{Z}_{\geq -n^2} + i) \cap \mathcal{D}(\lambda)$. *If* $\mathrm{Var}_\mathsf{n} \#_i \xrightarrow[n \to +\infty]{} +\infty$ *then:*

$$\frac{\#_i - \mathbb{E}_n \#_i}{\sqrt{\mathrm{Var}_\mathsf{n} \#_i}} \xrightarrow[n \to +\infty]{d} \mathcal{N}(0, 1).$$

Remark

The theorem was stated in a much more general setting.

Expectation and variance

Theorem (R. 21)

When $n \to +\infty$, one has:

$$\mathbb{E}_n x_i(\lambda) = O(1),$$

and:

$$\mathrm{Var}_\mathrm{n}\, x_i(\lambda) \sim \frac{4\sqrt{n}}{\pi e^2} \cot \frac{\pi}{2e}.$$

Corollary (R. 21)

Under the (Poissonised) Plancherel measure pl_n one has:

$$n^{-1/4} x_i(\lambda) \xrightarrow[n \to +\infty]{d} \mathcal{N}\left(0, \frac{4}{\pi e^2} \cot \frac{\pi}{2e}\right).$$

Joint asymptotics

We now use a multidimensional version of the central limit theorem (Soshnikov 2000).

Theorem (R. 21)

Under the (Poissonised) Plancherel measure pl_n *one has:*

$$e\sqrt{\frac{\pi}{2}}\left(\frac{x_i(\lambda)}{n^{1/4}}\right)_{i\in\mathbb{Z}/e\mathbb{Z}} \xrightarrow[n\to+\infty]{d} \mathcal{N}(0,B),$$

where $B = (b_{ij})$ *with* $b_{ij} := \cot\big(j-i+\frac{1}{2}\big)\frac{\pi}{e} - \cot\big(j-i-\frac{1}{2}\big)\frac{\pi}{e}$.

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In particular, if $\mu_0,\dots,\mu_{e-1}$ are the eigenvalues of B then:

$$\frac{e\pi}{2\sqrt{t}}\ell_e(\lambda) \xrightarrow[n\to+\infty]{d} \sum_{k=0}^{e-1}\Gamma\big(\tfrac{1}{2},\mu_k\big).$$

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Proposition (R. 21)

For all $k\in\{0,\dots,e-1\}$ *we have* $\mu_k = 2e\sin\frac{k\pi}{e}$.

Conclusion

Theorem (R. 21, main result)

Under the (Poissonised) Plancherel measure pl_n*, the size* $\ell_e(\lambda)$ *of the e-core satisfies:*

$$\frac{\pi}{4\sqrt{n}}\ell_e(\lambda) \xrightarrow[n\to+\infty]{d} \sum_{k=1}^{e-1} \Gamma\left(\tfrac{1}{2}, \sin\tfrac{k\pi}{e}\right)$$

(sum of mutually independent random variables).

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Lulov–Pittel (1999) and Ayyer–Sinha (2020) have shown that under the uniform measure on the set of partitions of n one has:

$$\frac{\pi}{\sqrt{n}}\ell_e(\lambda) \xrightarrow[n\to+\infty]{d} \Gamma\left(\tfrac{e-1}{2}, \sqrt{6}\right) = \sum_{k=1}^{e-1} \Gamma\left(\tfrac{1}{2}, \sqrt{6}\right).$$

In pictures

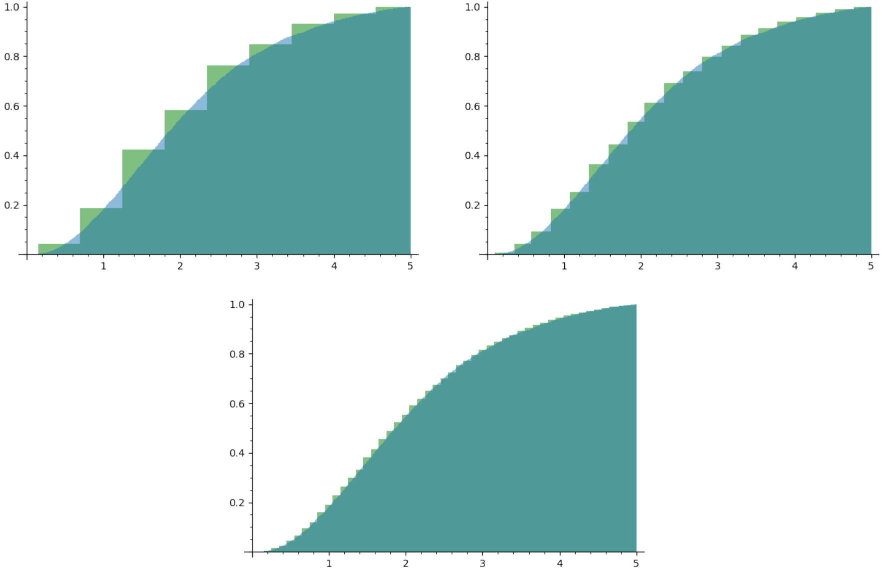

Figure: Convergence in distribution of $\frac{\pi}{4\sqrt{n}}\ell_e(\lambda)$ to $\sum_{k=1}^{e-1}\Gamma\left(\frac{1}{2}, \sin\frac{k\pi}{e}\right)$ for $e = 7$ and $n = 100, 500, 3000$.

The end

a	t	t	e	n	t	i	o	n
T	h	a	n	k				
y	o	u	r					
f	o	r						
y	o	u						
!								