# Skew cellularity of the Hecke algebra of $G(r, p, n)$

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1. Cellular algebras
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# Cellular algebras

Let $F$ be a field and $A$ a finite dimensional $F$-algebra.

**Definition (Graham–Lehrer 96)**

The algebra $A$ is cellular if there exists a poset $(\mathcal{P}, \rhd)$ with, for each $\lambda \in \mathcal{P}$,

- an indexing set $\mathcal{T}(\lambda)$
- elements $c_{\mathfrak{s},\mathfrak{t}} \in A$ for $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)$

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# Toy examples

## Example

The algebra $F[x]/(x^n)$ is cellular with the following data:

- $\mathcal{P} := \{0, \ldots, n-1\}$
- $\mathcal{T}(i) := \{i\}$
- $c_{i,i} := x^i$

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**Example**

The algebra $\mathrm{Mat}_{n\times n}(F)$ is cellular with the following data:

- $\mathcal{P} := \{n\}$ a singleton
- $\mathcal{T}(n) := \{1, \ldots, n\}$
- $c_{i,j} := E_{i,j}$ the elementary matrix with a 1 at position $(i, j)$ and a zero elsewhere.

**Proposition**

*Any semisimple algebra is cellular.*

# Ariki–Koike algebra

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*The Ariki–Koike algebra $\mathcal{H}_{r,n}$ is cellular, the poset being the set of $r$-partitions of $n$.*

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- From Murphy to Hu–Mathas, the order is always the dominance order on $r$-partitions.
- Webster and Bowman gave in fact many different cellular bases, associated with many different orders (generalising the dominance order). They intensively use a diagrammatic Cherednik algebra.

# Representation theory

Let $(A, \mathcal{P}, \rhd)$ be a cellular algebra. For all $\lambda \in \mathcal{P}$, we can construct a particular $A$-module $C^\lambda$, called cell module, together with a symmetric bilinear form $\phi_\lambda$.

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**Definition (Graham–Lehrer 96)**

For any $\lambda \in \mathcal{P}$, define the $A$-module $D^\lambda := C^\lambda / \mathrm{rad}\, \phi_\lambda$.

Let $\mathcal{P}_0 := \{\lambda \in \mathcal{P} : D^\lambda \neq \{0\}\}$, and define the decomposition matrix $D_A = (d_{\lambda,\mu})_{\lambda \in \mathcal{P}, \mu \in \mathcal{P}_0}$ by $d_{\lambda,\mu} := [C^\lambda, D^\mu]$.

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**Theorem (Graham–Lehrer 96)**

- *The family $\{D^\lambda : \lambda \in \mathcal{P}_0\}$ is a complete collection of non-isomorphic simple $A$-modules.*
- *For any $\lambda \in \mathcal{P}$ and $\mu \in \mathcal{P}_0$ we have $d_{\mu,\mu} = 1$ and $d_{\lambda,\mu} \neq 0 \iff \lambda \unrhd \mu$, in other words the decomposition matrix $D_A$ is upper unitriangular.*
- *The $A$-module $D^\lambda$ is self-dual.*

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# Aim

Let $(A, \mathcal{P}, \triangleright)$ be a cellular algebra and let $\sigma_A$ be an algebra automorphism of $A$.

## Question

What can we say about the subalgebra

$$A^\sigma = \{a \in A : \sigma_A(a) = a\} \quad ?$$

# Aim

Let $(A, \mathcal{P}, \rhd)$ be a cellular algebra and let $\sigma_A$ be an algebra automorphism of $A$.

**Question**

What can we say about the subalgebra

$$A^\sigma = \{a \in A : \sigma_A(a) = a\} \quad ?$$

- We will introduce the notion of skew cellular algebra.
- If $\sigma_A$ satisfies some conditions then $A^\sigma$ will be skew cellular.

# Skew cellular algebras

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such that:

- the set $\{c_{\mathfrak{s},\mathfrak{t}} : \lambda \in \mathcal{P}, \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)\}$ is an $F$-basis of $A$
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- for all $a \in A$, the product $c_{\mathfrak{s},\mathfrak{t}}a \in A$ decomposes in the basis $\{c_{\mathfrak{u},\mathfrak{v}}\}$ in a (particular) triangular fashion

# Skew cellular algebras

Let $A$ be a finite dimensional $F$-algebra and $\iota$ a poset involution of $\mathcal{P}$.

## Definition (Hu-Mathas-R. 21)

The algebra $A$ is skew-cellular if there exists a poset $(\mathcal{P}, \rhd)$ with, for each $\lambda \in \mathcal{P}$,

- an indexing set $\mathcal{T}(\lambda)$
- elements $c_{\mathfrak{s},\mathfrak{t}} \in A$ for $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)$
- a bijection $\iota_\lambda : \mathcal{T}(\lambda) \to \mathcal{T}(\iota\lambda)$ such that $\iota_{\iota\lambda} \circ \iota_\lambda = \mathrm{id}_{\mathcal{T}(\lambda)}$

such that:

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If $\iota = \mathrm{id}_{\mathcal{P}}$ and $\iota_\lambda = \mathrm{id}_{\mathcal{T}(\lambda)}$ then we recover Graham–Lehrer's definition of a cellular algebra.

Let $(A, \mathcal{P}, \rhd, \iota)$ be a skew cellular algebra. For all $\lambda \in \mathcal{P}$, again we can construct a particular $A$-module $C^\lambda$, called cell module, together with a bilinear form $\phi_\lambda$ (not necessarily symmetric).

**Definition (Graham–Lehrer 96, Hu-Mathas-R. 21)**

For any $\lambda \in \mathcal{P}$, define the $A$-module $D^\lambda := C^\lambda / \mathrm{rad}\, \phi_\lambda$.

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- the family $\{D^\lambda : \lambda \in \mathcal{P}_0\}$ is a complete collection of non-isomorphic simple $A$-modules;
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**Proposition (Hu-Mathas-R. 21)**

*For any $\lambda \in \mathcal{P}$ we have $D^{\iota\lambda} \simeq (D^\lambda)^*$. In particular:*

- $\lambda \in \mathcal{P}_0 \iff \iota\lambda \in \mathcal{P}_0$
- *if $\lambda = \iota\lambda$ then $D^\lambda$ is self-dual.*

# Shift automorphisms

## Definition (Hu-Mathas-R. 21)

Let $(A, \mathcal{P}, \rhd)$ be a cellular algebra. A shift automorphism of $A$ is a triple $(\sigma_A, \sigma_{\mathcal{P}}, \sigma_{\mathcal{T}})$ where:

- $\sigma_A$ is an algebra automorphism of $A$
- $\sigma_{\mathcal{P}}$ is a poset automorphism of $\mathcal{P}$
- $\sigma_{\mathcal{T}}$ is an automorphism of the set $\mathcal{T} = \amalg_{\lambda \in \mathcal{P}} \mathcal{T}(\lambda)$

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Let $\mathcal{P}_\sigma$ be the set of orbits of $\mathcal{P}$ under the action of $\langle\sigma_\mathcal{P}\rangle$.

## Lemma

*Let $\lambda, \mu \in \mathcal{P}$. The relation $\rhd_\sigma$ on $\mathcal{P}_\sigma$ defined by*

$$[\lambda] \rhd_\sigma [\mu] \iff \lambda \rhd \sigma^k\mu, \quad \text{for some } k \in \mathbb{Z},$$

*is (well-defined and) a partial order of $\mathcal{P}_\sigma$.*

# Subalgebra of fixed points

**Proposition (Hu-Mathas-R. 21)**

*Let $A$ be a cellular algebra with a shift automorphism $(\sigma_A, \sigma_{\mathcal{P}}, \sigma_{\mathcal{T}})$. Assume that $F$ contains a primitive $p$-th root of unity $\epsilon$, where $p$ is the order of $\sigma_A$. Then the subalgebra $A^\sigma$ of fixed points is a skew cellular algebra.*

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In more details:

- the poset is $\{([\lambda], k) : [\lambda] \in \mathcal{P}_\sigma, k \in \mathbb{Z}/\mathrm{o}_\lambda\mathbb{Z}\}$, where the order is induced by $\rhd_\sigma$
- the involution is $([\lambda], k) \longmapsto ([\lambda], -k)$
- the basis consists in elements of the form, with $\overline{\sigma}_A := \sum_{l=0}^{p-1} \sigma_A^l$,

$$c_{\mathfrak{s},\mathfrak{t}}^{(k)} := \sum_j \epsilon^{kj} \overline{\sigma}_A \big(c_{\mathfrak{s}, \sigma_{\mathcal{T}}^{j\mathrm{o}_\lambda} \mathfrak{t}}\big) \in A^\sigma$$

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## Corollary

*If $\sigma_A$ has order $p = 2$ then $A^\sigma$ is cellular.*

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# Complex reflection groups

**Definition**

- A complex reflection is a linear automorphism of $\mathbb{C}^n$ of finite order, different from identity, that fixes a hyperplane.
- A complex reflection group is a finite subgroup of $\mathrm{GL}(\mathbb{C}^n)$ spanned by complex reflections.

**Theorem (Shephard–Todd 54)**

*Irreducible complex reflection groups are divided into two families:*

- *an infinite family* $\{G(r,p,n)\}$ *with* $p \mid r$*;*
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- *34 exceptions.*

The group $G(r, p, n)$ is isomorphic to the group of $n \times n$ monomial matrices with entries in $\mu_r(\mathbb{C})$, where the product of all the non-zero entries lies in $\mu_{r/p}(\mathbb{C})$. It is a subgroup of index $p$ of $G(r, 1, n)$.

# Ariki–Koike algebras again

**Definition (Broué–Malle 93, Ariki–Koike 94)**

The Ariki–Koike $\mathcal{H}_{r,n}$ is the Hecke algebra of the complex reflection group $G(r, 1, n)$.

- Under some assumptions on the parameters, the algebra $\mathcal{H}_{r,n}$ is naturally equipped with an automorphism $\sigma$ of order $p$.

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- Following [Broué–Malle 93, Ariki 95, Broué-Malle-Rouquier 98], we can also define a Hecke algebra for $G(r, p, n)$, denoted by $\mathcal{H}_{r,p,n}$.

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**Proposition (Ariki 95)**

*The algebra $\mathcal{H}_{r,p,n}$ is the subalgebra of fixed points of $\mathcal{H}_{r,n}$ for the automorphism $\sigma$.*

# Skew cellularity of the Hecke algebra of $G(r, p, n)$

**Theorem (Geck 07)**

*If* $(r, p, n) \in \{(2, 2, n), (p, p, 2)\}$ *then* $\mathcal{H}_{r,p,n}$ *is cellular.*

Geck's result concerns in fact all Hecke algebras of finite Coxeter groups. His proof relies on Kazhdan–Lusztig theory.

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**Theorem (Hu-Mathas-R. 21)**

*For a particular cellular structure on* $\mathcal{H}_{r,n}$, *the automorphism* $\sigma$ *is a shift automorphism. In particular:*

- *the Hecke algebra* $\mathcal{H}_{r,p,n}$ *of* $G(r, p, n)$ *is skew cellular*
- *if* $p = 2$ *then* $\mathcal{H}_{2d,2,n}$ *is in fact cellular.*

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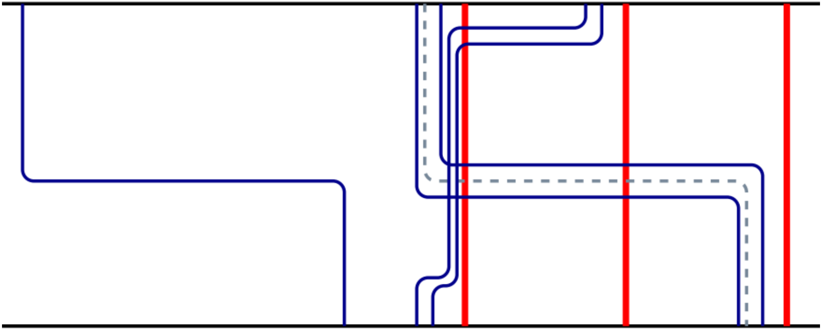
- An important point of the proof is to construct a very particular diagram inside the Cherednik algebra of Webster–Bowman.
- Our result is stronger that Geck's in the case $r = p = 2$ since we prove in general the graded skew cellularity.

# A glimpse of the diagrammatic Cherednik algebra

Here is an example of a diagram inside the diagrammatic Cherednik algebra of Webster–Bowman:



# Classification of the simple modules: Clifford theory

Let $A$ be a cellular algebra with a shift automorphism $(\sigma_A, \sigma_{\mathcal{P}}, \sigma_{\mathcal{T}})$ so that $A^\sigma$ is skew cellular. Let $D^\lambda$ (resp. $D^{\lambda,k}$) be the irreducible $A$-module (resp. $A^\sigma$-module) corresponding to $\lambda$.

**Proposition (Hu-Mathas-R. 21)**

*As A-modules we have*

$$^\sigma D_\lambda \simeq D_{\sigma_{\mathcal{P}}\lambda}$$

$$D^{\lambda,k}\Big\uparrow_{A^\sigma}^{A} \simeq \bigoplus_j D^{\sigma_{\mathcal{P}}^j\lambda}$$

*and as $A^\sigma$-modules we have*

$$\bigoplus_k D^{\lambda,k} \simeq D^\lambda\Big\downarrow_{A^\sigma}^{A}$$

$$D^{\lambda,k} \simeq {}^\tau D^{\lambda,k+1}$$

*where $\tau$ is the conjugation by an invertible element of* $\ker(\sigma_A - \epsilon)$.

- The same statement holds for the cell modules $C^\lambda$ and $C^{\lambda,k}$.
- We recover the existing classification of $\mathcal{H}_{r,p,n}$-modules.

# The end

Thank you!